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Paley–Wiener-Type Theorems for a Class of Integral Transforms

Vu Kim Tuan

Department of Mathematics and Computer Science, Faculty of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait
E-mail: vu@sci.kuniv.edu.kw

and

Ahmed I. Zayed[1]

Department of Mathematics, University of Central Florida, Orlando, Florida 32816
E-mail: zayed@pegasus.cc.ucf.edu

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A characterization of weighted $L_2(I)$ spaces in terms of their images under various integral transformations is derived, where I is an interval (finite or infinite). This characterization is then used to derive Paley–Wiener-type theorems for these spaces. Unlike the classical Paley–Wiener theorem, our theorems use real variable techniques and do not require analytic continuation to the complex plane. The class of integral transformations considered is related to singular Sturm–Liouville boundary-value problems on a half line and on the whole line. © 2002 Elsevier Science

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1. INTRODUCTION

The Paley–Wiener theorem [8, 9] gives a characterization of the space $L_2[-\delta, \delta]$ in terms of its image under the Fourier transformation by show-

[1] Current address: Department of Mathematical Sciences, DePaul University, 2320 N. Kenmore Avenue, Chicago, IL 60614. E-mail: azayed@math.depaul.edu.

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ing that $f \in L_2[-\delta, \delta]$ if and only if its Fourier transform $\hat{f}(\omega)$ can be continued analytically to the whole complex plane as an entire function of exponential type at most δ whose restriction to the real axis belongs to $L_2(R)$. Notwithstanding the strength of its statement, the proof of the theorem does not lend itself very naturally to other integral transformations. Alternative approaches using real analysis techniques have been developed to characterize the images of spaces of the form $L_2[I, d\rho]$, for some measure $d\rho$ and an interval I (finite or infinite) under various integral transformations, such as the Fourier [2, 3, 12, 18, 19], Mellin [16], Hankel [15], Y [14], and Airy transforms [17].

In all the work cited above, some ad-hoc techniques were used to suit the integral transformation under consideration. In this paper, we give a unified approach to handle a large class of integral transforms that includes not only most of the above transforms, but also many new transforms. This class of integral transforms, which was studied by one of the authors in a different context [20, 21, 23, 24] arises from two types of singular Sturm–Liouville problems: singular on a half line and singular on the whole line. The latter is more complicated, but it includes more interesting examples.

Let L be a differential operator, with a continuous spectrum Ω_1, and let $\phi(x, \lambda)$ be an eigenfunction with corresponding eigenvalue λ: $L\phi = -\lambda\phi$. In addition, suppose that T: $L_2(\Omega_1; d\rho_1) \to L_2(\Omega_2; d\rho_2)$

$$f(x) = (TF)(x) = \int_{\Omega_1} F(\lambda)\phi(x, \lambda)\, d\rho_1(\lambda)$$

is a unitary transformation

$$\int_{\Omega_2} |f(x)|^2\, d\rho_2(x) = \int_{\Omega_1} |F(\lambda)|^2\, d\rho_1(\lambda).$$

In that case, if $\lambda^n F(\lambda) \in L_2(\Omega_1; d\rho_1)$ we have

$$(L^n)f(x) = \int_{\Omega_1} (-\lambda)^n F(\lambda)\phi(x, \lambda)\, d\rho_1(\lambda),$$

and

$$\int_{\Omega_2} |(L^n f)(x)|^2\, d\rho_2(x) = \int_{\Omega_1} |\lambda^n F(\lambda)|^2\, d\rho_1(\lambda). \tag{1}$$

Raising both sides of (1) to the power $1/(2n)$ and taking the limit as $n \to \infty$ we get [13]

$$\lim_{n\to\infty} \|L^n f\|^{1/n}_{L_2(\Omega_2;\, d\rho_2)} = \sup_{\lambda \in \operatorname{supp} F} |\lambda|, \tag{2}$$

where $\operatorname{supp} F$ denotes the support of F, the smallest closed set, outside which the function F vanishes almost everywhere.

From (2) it is obvious that if

$$\lim_{n\to\infty} \|L^n f\|_{L_2(\Omega_2;\, d\rho_2)}^{1/n} < \infty, \tag{3}$$

then F has compact support. Hence, formula (2) plays a decisive role in studying integral transforms of functions with compact support. It can be shown [13] that under some "extra conditions" on f inequality (3) gives the necessary and sufficient condition for a function f to be a T-transform of a function $F \in L_2(\Omega_1; d\rho_1)$ with compact support. Formula (2) was first discovered in [2] for one-dimensional Fourier transforms with $L = \frac{d}{dx}$, and independently in [3, 12, 18, 19] for multidimensional Fourier transforms with L being the Laplacian or any other polynomial differential operator.

We shall look at the transform of a space of functions F that has the property $\lambda^n F(\lambda) \in L_2(\Omega_1; d\rho_1)$ for any n. In addition to inequality (3), the description of this space of functions gives us all the necessary "extra conditions" we need to characterize the space of functions with compact support under those transforms. As examples, we shall consider many classical transforms. For some of them (such as, the Weber transform) the complex Paley–Wiener theorem is unknown, and thus, our result is new, but for others (Fourier-sine and Fourier-cosine, Kontorovich–Lebedev, and Jacobi transforms) although their complex Paley–Wiener theorems are known, our theorems do not require the behavior of f in the complex plane, as the classical theorems do.

2. INTEGRAL TRANSFORMS RELATED TO SINGULAR STURM–LIOUVILLE PROBLEMS ON A HALF LINE

Consider the singular Sturm–Liouville problem on the half line

$$Ly := \frac{d^2 y}{dx^2} - q(x)y = -\lambda y, \qquad 0 \le x < \infty, \tag{4}$$

with

$$\begin{gathered} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad 0 \le \alpha < 2\pi, \\ |y(\infty)| < \infty, \end{gathered} \tag{5}$$

and q is assumed to be continuous on $R^+ = [0, \infty)$ and $q \in L_1(R^+)$.

Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be the solutions to Eq. (4) satisfying the initial conditions

$$\begin{aligned} \phi(0, \lambda) &= \sin\alpha, & \phi'(0, \lambda) &= -\cos\alpha, \\ \theta(0, \lambda) &= \cos\alpha, & \theta'(0, \lambda) &= \sin\alpha. \end{aligned} \tag{6}$$

Throughout the paper $\phi'(x, \lambda)$ will mean $\frac{\partial}{\partial x}\phi(x, \lambda)$. From the asymptotic expansions of $\phi(x, \lambda)$ and $\phi'(x, \lambda)$ as $x \to \infty$ [6]

$$\begin{aligned}\phi(x, \lambda) &= \mu(\lambda)\cos(\sqrt{\lambda}x) + \nu(\lambda)\sin(\sqrt{\lambda}x) + o(1),\\ \phi'(x, \lambda) &= -\lambda^{1/2}\mu(\lambda)\sin(\sqrt{\lambda}x) + \lambda^{1/2}\nu(\lambda)\sin(\sqrt{\lambda}x) + o(1),\end{aligned} \tag{7}$$

where

$$\mu(\lambda) = \sin\alpha - \lambda^{-1/2}\int_0^\infty \sin(\sqrt{\lambda}t)q(t)\phi(t, \lambda)\,dt,$$

$$\nu(\lambda) = -\lambda^{-1/2}\cos\alpha + \lambda^{-1/2}\int_0^\infty \cos(\sqrt{\lambda}t)q(t)\phi(t, \lambda)\,dt,$$

it is easy to see that $\phi(x, \lambda)$ and $\phi'(x, \lambda)$ are bounded as functions of x for $\lambda > 0$. The same is true for $\theta(x, \lambda)$ and $\theta'(x, \lambda)$.

It is known [11] that for any non-real λ, there exists a function $m(\lambda)$, analytic in the upper and lower half planes that are not necessarily analytic continuations of each other, so that

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\phi(x, \lambda),$$

as a function of x, is in $L_2(R^+)$ for non-real λ. Moreover,

$$\frac{1}{\pi}\lim_{\delta\to 0}\int_0^\lambda \Im\, m(u + i\delta)\,du = -\rho(\lambda),$$

where $\rho(\lambda)$ is a non-decreasing function and $\Im z$ means the imaginary part of z. It is a known fact [11] that if $f \in L_2(R^+)$, then

$$F(\lambda) = \int_0^\infty f(x)\phi(x, \lambda)\,dx \tag{8}$$

is well defined and belongs to $L_2(R, d\rho)$, and

$$f(x) = \int_{-\infty}^\infty F(\lambda)\phi(x, \lambda)\,d\rho(\lambda), \tag{9}$$

with

$$\|f\|_{L_2(R^+)} = \|F\|_{L_2(R, d\rho)}. \tag{10}$$

We shall assume as in many cases of interest that the support of $d\rho$ is R^+, and that $q \in C^\infty(R^+)$ and is bounded. In this case the transform (9) takes the form

$$f(x) = \int_0^\infty F(\lambda)\phi(x, \lambda)\,d\rho(\lambda), \tag{11}$$

and the Parseval equality (10) becomes

$$\|f\|_{L_2(R^+)} = \|F\|_{L_2(R^+, d\rho)}. \tag{12}$$

Restrictions on q to guarantee a continuous spectrum of the singular Sturm–Liouville problem (4)–(5) can be found in [6, 7, 11].

The object of this section is to study the integral transform (11), which we shall call *the ϕ-transform* of F.

LEMMA 1. *Let F be such that $\lambda^n F(\lambda) \in L_2(R^+, d\rho)$ for all $n = 0, 1, 2, \ldots$. A function f is the ϕ-transform of F as given by* (11) *if and only if*

(1-*i*) *f is infinitely differentiable on R^+,*
(1-*ii*) *$L^n f \in L_2(R^+)$ for all $n = 0, 1, 2, \ldots,$*
(1-*iii*) $\lim_{x\to 0+}\{\cos\alpha (L^n f)(x) + \sin\alpha \frac{d}{dx}(L^n f)(x)\} = 0$ *for all $n = 0, 1, 2, \ldots,$*
(1-*iv*) $\lim_{x\to\infty}(L^n f)(x) = \lim_{x\to\infty}\frac{d}{dx}(L^n f)(x) = 0$ *for all $n = 0, 1, 2, \ldots$.*

Proof. Necessity: Let $\lambda^n F(\lambda) \in L_2(R^+, d\rho)$ for all $n = 0, 1, 2, \ldots$. It is easy to see that $\lambda^n F(\lambda) \in L_1(R^+, d\rho)$. Indeed, applying formula (6.7.4) of [11, Chap. 6],

$$\int_0^\infty \frac{d\rho(\lambda)}{\lambda^2 + 1} < \infty,$$

and the Cauchy–Schwartz inequality, we get

$$\left|\int_0^\infty \lambda^n |F(\lambda)|\, d\rho(\lambda)\right|^2 \le \int_0^\infty \lambda^{2n}(1+\lambda^2)|F(\lambda)|^2\, d\rho(\lambda) \int_0^\infty \frac{d\rho(\lambda)}{\lambda^2+1} < \infty.$$

(1-i) Since $q(x)$ is infinitely differentiable, so is $\phi(x, \lambda)$. Moreover, $\frac{\partial^n \phi(x,\lambda)}{\partial x^n} = O(\lambda^{n/2})$ as $\lambda \to \infty$ [11]. Therefore,

$$f^{(n)}(x) = \int_0^\infty F(\lambda)\frac{\partial^n \phi(x,\lambda)}{\partial x^n} d\rho(\lambda)$$

exists for all $n = 0, 1, 2, \ldots$.

(1-ii) By applying the differential operator L^n to both sides of (11) we have

$$\begin{aligned}(L^n f)(x) &= \int_0^\infty F(\lambda) L^n \phi(x, \lambda)\, d\rho(\lambda) \\ &= (-1)^n \int_0^\infty \lambda^n F(\lambda)\phi(x,\lambda)\, d\rho(\lambda), \end{aligned} \tag{13}$$

and since by assumption $\lambda^n F(\lambda) \in L_2(R^+, d\rho)$ it follows that $L^n f \in L_2(R^+)$.

(1-iii) By taking the limit of (13) as $x \to 0+$, we have

$$\lim_{x\to 0+}(L^n f)(x) = \lim_{x\to 0+}(-1)^n \int_0^\infty \lambda^n F(\lambda)\phi(x,\lambda) d\rho(\lambda),$$

and by the Lebesgue dominated convergence theorem, we can take the limit inside the integral to obtain

$$\lim_{x\to 0+}(L^n f)(x) = (-1)^n \int_0^\infty \lambda^n F(\lambda)\phi(0,\lambda) d\rho(\lambda) = A_n \sin\alpha,$$

where $A_n = (-1)^n \int_0^\infty \lambda^n F(\lambda) d\rho(\lambda)$.

Likewise, if we differentiate (13) and then take the limit, we obtain

$$\lim_{x\to 0+}\frac{d}{dx}(L^n f)(x) = \lim_{x\to 0+}(-1)^n\int_0^\infty \lambda^n F(\lambda)\phi'(x,\lambda)\,d\rho(\lambda)$$
$$= (-1)^n\int_0^\infty \lambda^n F(\lambda)\phi'(0,\lambda)d\rho(\lambda) = -A_n\cos\alpha.$$

Hence,

$$\lim_{x\to 0+}\left\{\cos\alpha\,(L^n f)(x) + \sin\alpha\,\frac{d}{dx}(L^n f)(x)\right\} = 0.$$

(1-iv) Because q is bounded, $q L^n f \in L_2(R^+)$, and therefore, $\frac{d^2}{dx^2}(L^n f)\times(x) = (L^{n+1}f)(x) + q(x)(L^n f)(x) \in L_2(R^+)$. Now the fact that $(L^n f)(x)$, $(L^{n+1}f)(x) \in L_2(R^+)$ yields $\frac{d}{dx}(L^n f)(x) \in L_2(R^+)$. For, if we denote by $\hat{f}_n(\omega)$ the Fourier transform

$$\hat{f}_n(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_n(x)e^{-i\omega x}dx$$

of $f_n(x)$ that is an infinitely differentiable extension of $(L^n f)(x)$ from R^+ over R with a support bounded below, then the Fourier transform of $\frac{d}{dx}f_n(x)$ and $\frac{d^2}{dx^2}f_n(x)$ are $i\omega\hat{f}_n(\omega)$ and $(i\omega)^2\hat{f}_n(\omega)$, respectively. To show that $\frac{d}{dx}(L^n f)(x) \in L_2(R^+)$ it suffices to prove that $\frac{d}{dx}f_n(x) \in L_2(R)$, which is equivalent to showing that $\omega\hat{f}_n(\omega) \in L_2(R)$. But this follows from the Cauchy–Schwarz inequality

$$\left(\int_{-\infty}^{\infty}\omega^2\left|\hat{f}_n(\omega)\right|^2 d\omega\right)^2 \le \left(\int_{-\infty}^{\infty}\omega^4\left|\hat{f}_n(\omega)\right|^2 d\omega\right)\left(\int_{-\infty}^{\infty}\left|\hat{f}_n(\omega)\right|^2 d\omega\right) < \infty.$$

The last inequality holds since $\hat{f}_n(\omega)$ and $\omega^2\hat{f}_n(\omega)$ are in $L_2(R)$.

Now we have

$$2\int_0^x (L^n f)(t)\frac{d}{dt}(L^n f)(t)\,dt = (L^n f)^2(x) - \lim_{x\to 0+}(L^n f)^2(x)$$
$$= (L^n f)^2(x) - (A_n\sin\alpha)^2.$$

But since $(L^n f)(t)\frac{d}{dt}(L^n f)(t) \in L_1(R^+)$, the limit of the right-hand side exists as $x \to \infty$. Consequently, $\lim_{x\to\infty}(L^n f)^2(x)$ exists. But $(L^n f)(x) \in L_2(R^+)$, then the limit must be zero:

$$\lim_{x\to\infty}(L^n f)(x) = 0.$$

Similarly, from the relation

$$2\int_0^x \frac{d}{dt}(L^n f)(t)\frac{d^2}{d^2 t}(L^n f)(t)\,dt = \left(\frac{d}{dx}(L^n f)(x)\right)^2 - \lim_{x\to 0+}\left(\frac{d}{dx}(L^n f)(x)\right)^2$$
$$= \left(\frac{d}{dx}(L^n f)(x)\right)^2 - (A_n\cos\alpha)^2,$$

and $\left(\frac{d}{dx}(L^n f)(x)\right)^2 \in L_1(R^+)$ it follows

$$\lim_{x\to\infty} \frac{d}{dx}(L^n f)(x) = 0.$$

Sufficiency: Let f satisfy the conditions (1-i)–(1-iv) of the lemma. Since $f \in L_2(R^+)$ (property (1-ii) with $n = 0$), there is $F \in L_2(R^+, d\rho)$ such that

$$F(\lambda) = \int_0^\infty f(x)\phi(x, \lambda)\, dx.$$

Then,

$$(-\lambda)^n F(\lambda) = \int_0^\infty f(x)(-\lambda)^n \phi(x, \lambda)\, dx = \int_0^\infty f(x) L^n \phi(x, \lambda)\, dx.$$

We use induction on n to show that

$$(-\lambda)^n F(\lambda) = \int_0^\infty (L^n f)(x)\phi(x, \lambda)\, dx.$$

It is true for $n = 0$. Assume that the formula holds for n. Then integration by parts yields

$$\begin{aligned}(-\lambda)^{n+1} F(\lambda) &= \int_0^\infty (L^n f)(x) L\phi(x, \lambda)\, dx \\ &= \left[(L^n f)(x)\phi'(x, \lambda) - \frac{d}{dx}(L^n f)(x)\phi(x, \lambda)\right]\Big|_0^\infty \\ &\quad + \int_0^\infty (L^{n+1} f)(x)\phi(x, \lambda)\, dx.\end{aligned}$$

But since

$$\phi(0, \lambda) = \sin\alpha, \quad \phi'(0, \lambda) = -\cos\alpha,$$

from property (1-iii) it is easy to see that

$$(L^n f)(0)\phi'(0, \lambda) - \frac{d}{dx}(L^n f)(0)\phi(0, \lambda) = 0,$$

and, by property (1-iv) and the boundedness of $\phi(x, \lambda)$ and $\phi'(x, \lambda)$ as $x \to \infty$, the same term is zero at $x = \infty$. Therefore,

$$(-\lambda)^{n+1} F(\lambda) = \int_0^\infty (L^{n+1} f)(x)\phi(x, \lambda)\, dx,$$

and since $L^{n+1} f \in L_2(R^+)$, it follows that $\lambda^{n+1} F(\lambda) \in L_2(R^+, d\rho)$. ∎

Remark 1. Lemma 1 includes Sturm–Liouville problems on a finite interval (a, b) with q having a singularity at one end-point only. On the other hand, Lemma 1 does not hold if $q(x)$ has a singularity at $x = 0$. This case, which is equivalent to a Sturm–Liouville problem on the whole line, will be treated in Section 3. An analysis of the proof of Lemma 1 suggests that the condition (1-iii) in this case should be replaced by

(1-iii′) $\lim_{x\to 0+}\{\phi'(x,\lambda)(L^n f)(x) - \phi(x,\lambda)\frac{d}{dx}(L^n f)(x)\} = \lim_{x\to 0+} W_x(L^n f, \phi) = 0$ for all $n = 0, 1, 2, \ldots,$

where W is the Wronskian. Also, condition (1-iii′) should be used in case $\sin\alpha$ or $\cos\alpha$ equals zero.

LEMMA 2. *Let* $\lambda^n F_j(\lambda) \in L_2(R, d\rho_j)$ *for all* $n = 0, 1, 2, \ldots,$ *and* $j = 1, 2, \ldots, m$. *Then*

$$\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)} = \max_{1\le j\le m}\sup_{\lambda\in\operatorname{supp}F_j}|\lambda|.$$

Proof. First let F_j have compact support: $\max_{1\le j\le m}\sup_{\lambda\in\operatorname{supp}F_j}|\lambda| = \delta < \infty$. Then

$$\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda) = \int_{-\delta}^{\delta}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda) \le \delta^{2n}\int_{-\delta}^{\delta}|F_j(\lambda)|^2\,d\rho_j(\lambda).$$

Hence,

$$\limsup_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)}$$
$$\le \delta\limsup_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\delta}^{\delta}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)} = \delta.$$

On the other hand, since δ is the supremum of the supports of F_j, for any $\epsilon > 0$,

$$\sum_{j=1}^{m}\int_{\delta-\epsilon<|\lambda|<\delta}|F_j(\lambda)|^2\,d\rho_j(\lambda) > 0.$$

Therefore,

$$\liminf_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)}$$
$$\ge \liminf_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{\delta-\epsilon<|\lambda|<\delta}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)}$$
$$\ge (\delta-\epsilon)\liminf_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{\delta-\epsilon<|\lambda|<\delta}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)} = \delta-\epsilon.$$

Because $\epsilon > 0$ is arbitrary, we obtain

$$\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)} = \delta.$$

Let us assume at least one of the F_j's has unbounded support. Then for any N large enough

$$\sum_{j=1}^{m}\int_{|\lambda|>N}|F_j(\lambda)|^2\,d\rho_j(\lambda) > 0.$$

Consequently,

$$\begin{aligned}
&\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)}\\
&\quad\geq \lim_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{|\lambda|>N}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)}\\
&\quad\geq N\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{|\lambda|>N}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)} = N.
\end{aligned}$$

Because N is arbitrary, we obtain

$$\lim_{n\to\infty}\left\{\sum_{j=1}^{m}\int_{-\infty}^{\infty}\lambda^{2n}|F_j(\lambda)|^2\,d\rho_j(\lambda)\right\}^{1/(2n)} = \infty.$$

∎

THEOREM 1. *A function f is the ϕ-transform* (11) *of a function $F \in L_2(R^+, d\rho)$ with compact support if and only if f satisfies conditions* (1-i)–(1-iv) *of Lemma* 1 *and*

$$\lim_{n\to\infty}\|L^n f\|_{L_2(R^+)}^{1/n} < \infty. \tag{14}$$

Proof. From the relation

$$(L^n f)(x) = (-1)^n\int_{-\infty}^{\infty}\lambda^n F(\lambda)\phi(x,\lambda)d\rho(\lambda),$$

and the Parseval equation (12) we have

$$\|L^n f\|_{L_2(R)}^2 = \int_{-\infty}^{\infty}\lambda^{2n}|F(\lambda)|^2\,d\rho(\lambda).$$

Let $F \in L_2(R^+, d\rho)$ have compact support. Then $\lambda^n F(\lambda) \in L_2(R^+, d\rho)$ for all $n = 0, 1, 2, \ldots$. Consequently, by Lemma 1, f satisfies conditions (1-i)–(1-iv), and by Lemma 2

$$\lim_{n\to\infty} \|L^n f\|_{L_2(R^+)}^{1/n} = \left\{\int_{-\infty}^{\infty} \lambda^{2n}|F(\lambda)|^2\, d\rho(\lambda)\right\}^{1/(2n)} = \sup_{\lambda\in\operatorname{supp}\, F} \lambda < \infty.$$

Conversely, let f satisfy conditions (1-i)–(1-iv) of Lemma 1 and (14). By Lemma 1, f is the ϕ-transform (11) of a function F such that $\lambda^n F(\lambda) \in L_2(R^+, d\rho)$ for all $n = 0, 1, 2, \ldots$. By Lemma 2 and Eq. (14) we have

$$\sup_{\lambda\in\operatorname{supp}\, F} \lambda = \left\{\int_{-\infty}^{\infty} \lambda^{2n}|F(\lambda)|^2\, d\rho(\lambda)\right\}^{1/(2n)} = \lim_{n\to\infty} \|L^n f\|_{L_2(R^+)}^{1/n} < \infty.$$

Hence, F has compact support. ∎

EXAMPLE 1 (The Weber transform). Consider the Bessel differential equation on the half line $[a, \infty)$, $a > 0$:

$$y'' - \frac{\nu^2 - 1/4}{x^2} y = -\lambda y, \qquad x \in [a, \infty), \quad \nu > -\frac{1}{2}. \tag{15}$$

Let $\alpha = 0$ and $\phi(x, \lambda)$ be the solution to Eq. (15) under the initial conditions

$$\phi(a, \lambda) = 0, \qquad \phi'(a, \lambda) = -1.$$

We have from [11]

$$\begin{aligned}\phi(x, \lambda) &= \frac{1}{2}\pi\sqrt{ax}[J_\nu(xs)Y_\nu(as) - Y_\nu(xs)J_\nu(as)],\\ d\rho(\lambda) &= \frac{4}{\pi^2 a}\frac{s\, ds}{J_\nu^2(as) + Y_\nu^2(as)},\end{aligned}$$

where $\lambda = s^2$, and correspondingly, we obtain the pair of Weber transforms

$$\begin{aligned}F(s) &= \int_a^\infty \sqrt{x}[J_\nu(xs)Y_\nu(as) - Y_\nu(xs)J_\nu(as)]f(x)\, dx,\\ f(x) &= \int_0^\infty \sqrt{x}\, \frac{[J_\nu(xs)Y_\nu(as) - Y_\nu(xs)J_\nu(as)]}{J_\nu^2(as) + Y_\nu^2(as)}\, sF(s)\, ds,\end{aligned} \tag{16}$$

with the Parseval equation

$$\int_a^\infty |f(x)|^2\, dx = \int_0^\infty \frac{\pi s|F(s)|^2}{J_\nu^2(as) + Y_\nu^2(as)}\, ds.$$

Applying Theorem 1 and Lemma 1 to the Weber transform (16) with the conditions at 0 being replaced by conditions at a we get

COROLLARY 1. *Let* $L := \frac{d^2}{dx^2} - \frac{\nu^2-1/4}{x^2}$. *A function* f *is the Weber transform* (16) *of a function*

$$F(s) \in L_2\left(R^+, \frac{s\,ds}{J_\nu^2(as) + Y_\nu^2(as)}\right)$$

with compact support if and only if, for any $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
&f \text{ is infinitely differentiable on } (a, \infty);\\
&L^n f \in L_2(a, \infty);\\
&\lim_{x\to\infty} (L^n f)(x) = \lim_{x\to\infty} \frac{d}{dx}(L^n f)(x) = 0;\\
&\lim_{x\to a+} (L^n f)(x) = 0, \quad \lim_{x\to a+} \frac{d}{dx}(L^n f)(x) \text{ exists},\\
&\lim_{n\to\infty} \|L^n f\|_{L_2(a,\infty)}^{1/n} < \infty.
\end{aligned}$$

EXAMPLE 2 (The Fourier-sine transform). Consider the boundary-value problem on the half line:

$$y'' = -\lambda y,\ \ x \in [0, \infty),\ \ y(0) = 0,\ \ |y(\infty)| < \infty.$$

In this case

$$\phi(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}},\ \ d\rho(\lambda) = \frac{\sqrt{\lambda}}{\pi}\,d\lambda \text{ on } [0, \infty).$$

Therefore,

$$\widetilde{F}(\lambda) = \int_0^\infty f(x)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\,dx,$$

and

$$f(x) = \frac{1}{\pi}\int_0^\infty \widetilde{F}(\lambda)\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}\sqrt{\lambda}\,d\lambda = \frac{1}{\pi}\int_0^\infty \widetilde{F}(\lambda)\sin\sqrt{\lambda}x\,d\lambda.$$

Making the change of variable $\lambda = s^2$ and $F(s) = \sqrt{\lambda}\widetilde{F}(\lambda)$, we obtain the pair of Fourier-sine transforms

$$\begin{aligned}
F(s) &= \int_0^\infty f(x)\sin sx\,dx,\\
f(x) &= \frac{2}{\pi}\int_0^\infty F(s)\sin sx\,ds.
\end{aligned} \tag{17}$$

Now the fact that $f^{(2n)}(x), f^{(2n+2)}(x) \in L_2(R)$ yields $f^{(2n+1)}(x) \in L_2(R)$. Hence, condition (1-ii) is equivalent to

$$f^{(n)}(x) \in L_2(R^+) \quad \text{for any } n = 0, 1, \ldots. \tag{18}$$

Because $\alpha = 0$, condition (1-iii) will be satisfied if

$$\lim_{x\to 0+} f^{(2n)}(x) = 0, \tag{19}$$

$$\lim_{x\to 0+} f^{(2n+1)}(x) \quad \text{exists.} \tag{20}$$

Condition (20) can be dropped. Indeed, because of (18) the integral $\int_x^1 f^{(2n+1)}(t)f^{(2n+2)}(t)\,dt$ converges; therefore, integration by parts yields

$$2\int_x^1 f^{(2n+1)}(t)f^{(2n+2)}(t)\,dt = [f^{(2n+1)}(1)]^2 - [f^{(2n+1)}(x)]^2$$

and after that taking a limit as $x \to 0+$, we obtain that $\lim_{x\to 0+} f^{(2n+1)}(x)$ exists for any n, and (20) is therefore automatically satisfied. At last, condition (1-iv) takes the form

$$\lim_{x\to\infty} f^{(n)}(x) = 0. \tag{21}$$

Hence, for the Fourier-sine transform (17) we obtain

COROLLARY 2. *A function f is the Fourier-sine transform* (17) *of a function $F \in L_2(R^+)$ with compact support if and only if f is infinitely differentiable, satisfying conditions* (18), (19), *and* (21) *for any $n = 0, 1, 2, \ldots,$ and in addition,*

$$\lim_{n\to\infty} \|f^{(n)}(x)\|_{L_2(R^+)}^{1/n} < \infty.$$

EXAMPLE 3 (The Fourier-cosine transform). Consider the same boundary-value problem, but with a different initial condition:

$$y'' = -\lambda y, \qquad x \in [0, \infty), \qquad y'(0) = 0.$$

We get

$$\phi(x, \lambda) = \cos\sqrt{\lambda}x, \qquad d\rho(\lambda) = \frac{1}{\sqrt{\lambda}\pi}\,d\lambda.$$

Therefore,

$$F(\lambda) = \int_0^\infty f(x)\cos\sqrt{\lambda}x\,dx,$$

and

$$f(x) = \frac{1}{\pi}\int_0^\infty F(\lambda)\frac{\cos\sqrt{\lambda}x}{\sqrt{\lambda}}\,d\lambda.$$

Making the change of variable $\lambda = s^2$ we obtain the pair of Fourier-cosine transforms

$$
\begin{aligned}
F(s) &= \int_0^\infty f(x)\cos sx\,dx, \\
f(x) &= \frac{2}{\pi}\int_0^\infty F(s)\cos sx\,ds.
\end{aligned}
\tag{22}
$$

Similar arguments to that of Example 2 lead to the following corollary for the Fourier-cosine transform (22):

COROLLARY 3. *A function f is the Fourier-cosine transform* (22) *of a function $F \in L_2(R^+)$ with compact support if and only if, for any $n = 0, 1, 2, \ldots,$*

$$
\begin{aligned}
&f \text{ is infinitely differentiable on } R^+; \\
&f^{(n)}(x) \in L_2(R^+); \\
&\lim_{x\to\infty} f^{(n)}(x) = \lim_{x\to 0+} f^{(2n+1)}(x) = 0; \\
&\lim_{n\to\infty} \|f^{(n)}(x)\|_{L_2(R^+)}^{1/n} < \infty.
\end{aligned}
$$

3. INTEGRAL TRANSFORMS RELATED TO SINGULAR STURM–LIOUVILLE PROBLEMS ON THE WHOLE LINE

Now we study integral transforms related to singular Sturm–Liouville problems on the whole line. But it should be mentioned that the singular case on the whole line includes as a special case the case of a singular Sturm–Liouville problem on the half line $(0, \infty)$ with $q(x)$ having a singularity at $x = 0$ and the case of a finite interval (a, b) with $q(x)$ having singularities at $x = a$ and $x = b$.

Let us consider the singular Sturm–Liouville problem

$$
Ly := \frac{d^2y}{dx^2} - q(x)y = -\lambda y, \quad -\infty < x < \infty, \quad |y(\pm\infty)| < \infty, \tag{23}
$$

with q being infinitely differentiable on R such that the spectrum of the singular Sturm–Liouville problem (23) is R. Let $\phi(x, \lambda)$ and $\theta(x, \lambda)$ be the solutions to Eq. (23) such that

$$
\begin{aligned}
\phi(0, \lambda) &= 0, \qquad & \phi'(0, \lambda) &= -1, \\
\theta(0, \lambda) &= 1, \qquad & \theta'(0, \lambda) &= 0.
\end{aligned}
$$

so that the Wronskian $W_x(\phi, \theta) = \phi(x, \lambda)\theta'(x, \lambda) - \phi'(x, \lambda)\theta(x, \lambda) = 1$.

Similar to the singular case on the half line, there exist two functions $m_1(\lambda)$ and $m_2(\lambda)$ analytic in the upper half plane such that, as a function of x,

$$\psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda)$$

is in $L_2(-\infty,0)$, and

$$\psi_2(x,\lambda) = \theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda)$$

is in $L_2(0,\infty)$, for λ with $\Im\lambda > 0$. Moreover, there exist two non-decreasing functions $\xi(\lambda)$, $\zeta(\lambda)$, and a function of bounded variation $\eta(\lambda)$ such that

$$\frac{1}{\pi}\lim_{\delta\to 0+}\int_0^\lambda \Im\frac{1}{m_1(u+i\delta)-m_2(u+i\delta)}\,du = -\xi(\lambda),$$
$$\frac{1}{\pi}\lim_{\delta\to 0+}\int_0^\lambda \Im\frac{m_1(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\,du = -\eta(\lambda),$$
$$\frac{1}{\pi}\lim_{\delta\to 0+}\int_0^\lambda \Im\frac{m_1(u+i\delta)m_2(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\,du = -\zeta(\lambda).$$

The functions $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are infinitely differentiable and $\frac{d^n}{dx^n}\theta(x,\lambda) = O(\lambda^{n/2})$, $\frac{d^n}{dx^n}\phi(x,\lambda) = O(\lambda^{n/2})$ for all $n = 0,1,2,\ldots$. Moreover,

$$\lim_{x\to\pm\infty}\int_a^b \theta(x,\lambda)\,d\xi(\lambda) = 0, \qquad \lim_{x\to\pm\infty}\int_a^b \theta'(x,\lambda)\,d\xi(\lambda) = 0,$$
$$\lim_{x\to\pm\infty}\int_a^b \phi(x,\lambda)\,d\zeta(\lambda) = 0, \qquad \lim_{x\to\pm\infty}\int_a^b \phi'(x,\lambda)\,d\zeta(\lambda) = 0,$$

for any a, b, $-\infty < a < b < \infty$.

From the corresponding relations for ϕ and θ, we deduce

$$\frac{d^n}{dx^n}\psi_j(x,\lambda) = O(\lambda^{n/2}) \qquad \text{for all } n = 0,1,2,\ldots, \tag{24}$$

$$\lim_{x\to\pm\infty}\int_a^b \psi_j(x,\lambda)\,d\rho_j(\lambda) = 0, \tag{25}$$

$$\lim_{x\to\pm\infty}\int_a^b \psi_j'(x,\lambda)\,d\rho_j(\lambda) = 0, \tag{26}$$

for some measures $d\rho_j$, $j = 1,2$, and any a, b, $-\infty < a < b < \infty$, where $d\rho_j$ is either $d\xi$ or $d\zeta$.

It is a known fact [11] that if $f \in L_2(R)$, then

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)\phi(x,\lambda)\,dx \tag{27}$$

belongs to $L_2(R, d\zeta)$, and

$$E(\lambda) = \int_{-\infty}^{\infty} f(x)\theta(x, \lambda)\, dx \tag{28}$$

belongs to $L_2(R, d\xi)$. The following inverse formula

$$\begin{aligned} f(x) = \int_{-\infty}^{\infty} E(\lambda)\theta(x, \lambda)\, d\xi(\lambda) + \int_{-\infty}^{\infty} F(\lambda)\theta(x, \lambda)\, d\eta(\lambda) \\ + \int_{-\infty}^{\infty} E(\lambda)\phi(x, \lambda)\, d\eta(\lambda) + \int_{-\infty}^{\infty} F(\lambda)\phi(x, \lambda)\, d\zeta(\lambda) \end{aligned} \tag{29}$$

holds, and in addition

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |E(\lambda)|^2\, d\xi(\lambda) + 2\int_{-\infty}^{\infty} E(\lambda)F(\lambda)\, d\eta(\lambda) \\ + \int_{-\infty}^{\infty} |F(\lambda)|^2\, d\zeta(\lambda). \end{aligned} \tag{30}$$

In most cases of interest these formulas take simpler forms, the most important of which are the following:

(A) $m_1(\lambda)$ tends to a real limit as $\Im\lambda \to 0+$. Then we have

$$G(\lambda) = \int_{-\infty}^{\infty} f(x)\psi_1(x, \lambda)\, dx, \tag{31}$$

and

$$f(x) = \int_{-\infty}^{\infty} G(\lambda)\psi_1(x, \lambda)\, d\xi(\lambda), \tag{32}$$

and the Parseval formula (30) looks simpler

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |G(\lambda)|^2\, d\xi(\lambda).$$

(B) If

$$\begin{aligned} \lim_{\delta\to 0+}\lim_{R\to\infty}\int_{-R+i\delta}^{R+i\delta} \frac{\psi_2(x, \lambda)\psi_1(y, \lambda)}{m_1(\lambda) - m_2(\lambda)}\, d\lambda \\ = \lim_{\delta\to 0+}\lim_{R\to\infty}\int_{-R+i\delta}^{R+i\delta} \frac{\psi_1(x, \lambda)\psi_2(y, \lambda)}{m_1(\lambda) - m_2(\lambda)}\, d\lambda, \end{aligned}$$

then

$$F(\lambda) = \int_{-\infty}^{\infty} \psi_2(y, \lambda)f(y)\, dy,$$

$$f(x) = \int_{-\infty}^{\infty} \psi_1(x, \lambda)F(\lambda)\, d\xi(\lambda),$$

and the Parseval equation is

$$\|f\|_{L_2(R)} = \|F\|_{L_2(R,d\xi)}.$$

In some cases of interest, $\psi_2(x, \lambda)$ and $d\xi$ are even or odd in λ and

$$\psi_1(x, \lambda) \pm \psi_1(x, -\lambda) = \psi_2(x, \lambda).$$

In those cases we obtain more symmetric formulas

$$F(\lambda) = \int_{-\infty}^{\infty} \psi_2(y, \lambda) f(y)\, dy,$$

$$f(x) = \int_{0}^{\infty} \psi_2(x, \lambda) F(\lambda)\, d\xi(\lambda).$$

(C) q is an even function. In this case $d\eta(\lambda) = 0$, and formula (29) takes the form

$$f(x) = \int_{-\infty}^{\infty} E(\lambda)\theta(x, \lambda)\, d\xi(\lambda) + \int_{-\infty}^{\infty} F(\lambda)\phi(x, \lambda)\, d\zeta(\lambda), \tag{33}$$

with formulas (27) and (28) remaining the same. The Parseval formula (30) in this case becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |E(\lambda)|^2\, d\xi(\lambda) + \int_{-\infty}^{\infty} |F(\lambda)|^2\, d\zeta(\lambda).$$

Cases (A), (B), and (C) are special cases of the pair of transforms,

$$F_j(\lambda) = \int_{-\infty}^{\infty} \psi_j(x, \lambda) f(x)\, dx, \;\; j = 1, \ldots, m, \tag{34}$$

$$f(x) = \sum_{j=1}^{m} \int_{-\infty}^{\infty} \psi_j(x, \lambda) F_j(\lambda)\, d\rho_j(\lambda), \tag{35}$$

that satisfies the Parseval equation

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \sum_{j=1}^{m} \int_{-\infty}^{\infty} |F_j(\lambda)|^2\, d\rho_j(\lambda).$$

Here m is either 1 or 2, $\rho_j(\lambda)$ are non-decreasing functions and $\psi_j(x, \lambda)$, $j = 1, \ldots, m$, are linearly independent solutions of Eq. (23) satisfying properties (24)–(26).

If $m = 1$ we get cases (A) and (B), and if $m = 2$, we obtain case (C).

For the pair of transforms (34) and (35) we have the following result:

LEMMA 3. *A function f is the transform* (35) *of F_j such that $\lambda^n F_j(\lambda) \in L_2(R, d\rho_j)$ for all $n = 0, 1, 2, \ldots$, and $j = 1, \ldots, m$, if and only if*

(3-*i*) *f is infinitely differentiable on R,*
(3-*ii*) *$L^n f \in L_2(R)$ for all $n = 0, 1, 2, \ldots,$*
(3-*iii*) $\lim_{x\to\pm\infty} (L^n f)(x) = \lim_{x\to\pm\infty} \frac{d}{dx}(L^n f)(x) = 0$ *for all* $n = 0, 1, 2, \ldots.$

Proof. Necessity: Let $\lambda^n F_j(\lambda) \in L_2(R, d\rho_j)$ for all $n = 0, 1, 2, \ldots,$ and $j = 1, \ldots, m$. By similarity with the proof of the fact $\lambda^n F(\lambda) \in L_1(R^+, d\rho)$ in Lemma 1, using formula (7.1.1), of [6, Chap. 7],

$$\int_{-\infty}^{\infty} \frac{d\xi(\lambda)}{\lambda^2 + 1} < \infty, \qquad \int_{-\infty}^{\infty} \frac{d\zeta(\lambda)}{\lambda^2 + 1} < \infty,$$

it is easy to show that $\lambda^n F_j(\lambda) \in L_1(R, d\rho_j)$ for any $n = 0, 1, 2, \ldots$.

(3-i) Since $q(x)$ is infinitely differentiable, so is $\psi_j(x, \lambda)$. Moreover, by (24),

$$f^{(n)}(x) = \sum_{j=1}^{m} \int_{-\infty}^{\infty} F_j(\lambda) \frac{\partial^n \psi_j(x, \lambda)}{\partial x^n}\, d\rho_j(\lambda)$$

exists for all $n = 0, 1, 2, \ldots$.

(3-ii) By applying the differential operator L^n to both sides of (35) we have

$$\begin{aligned}(L^n f)(x) &= \sum_{j=1}^{m} \int_{-\infty}^{\infty} F_j(\lambda) L^n \psi_j(x, \lambda)\, d\rho_j(\lambda) \\ &= \sum_{j=1}^{m} (-1)^n \int_{-\infty}^{\infty} \lambda^n F_j(\lambda) \psi_j(x, \lambda)\, d\rho_j(\lambda),\end{aligned}$$

and since by assumption $\lambda^n F_j(\lambda) \in L_2(R, d\rho_j)$ it follows that $L^n f \in L_2(R)$.

(3-iii) In view of the properties (25)–(26), by the Hobson lemma [10],

$$\lim_{x \to \pm\infty} \int_a^b g(\lambda) \psi_j(x, \lambda)\, d\rho_j(\lambda) = \lim_{x \to \pm\infty} \int_a^b g(\lambda) \psi_j'(x, \lambda)\, d\rho_j(\lambda) = 0,$$

for any $a, b, -\infty < a < b < \infty$, and any $g(\lambda) \in L_1(a, b)$. Because $\lambda^n F_j(\lambda) \in L_1(R, d\rho_j)$ for any $n = 0, 1, 2, \ldots,$ a similar discussion as in the proof of Lemma 1 shows that

$$\lim_{x \to \pm\infty} (L^n f)(x) = \lim_{x \to \pm\infty} \sum_{j=1}^{m} (-1)^n \int_{-\infty}^{\infty} \lambda^n F_j(\lambda) \psi_j(x, \lambda)\, d\rho_j(\lambda) = 0,$$

$$\lim_{x \to \pm\infty} \frac{d}{dx} (L^n f)(x) = \lim_{x \to \pm\infty} \sum_{j=1}^{m} (-1)^n \int_{-\infty}^{\infty} \lambda^n F_j(\lambda) \psi_j'(x, \lambda)\, d\rho_j(\lambda) = 0.$$

Sufficiency: Let f satisfy the conditions (3-i)–(3-iii) of the lemma. Since $f \in L_2(R)$ (property (3-ii) with $n = 0$), there are $F_j \in L_2(R, d\rho_j)$, $j = 1, \ldots, m$, such that

$$F_j(\lambda) = \int_{-\infty}^{\infty} f(x) \psi_j(x, \lambda)\, dx.$$

Then,

$$(-\lambda)^n F_j(\lambda) = \int_{-\infty}^{\infty} f(x)(-\lambda)^n \psi_j(x,\lambda)\,dx = \int_{-\infty}^{\infty} f(x) L^n \psi_j(x,\lambda)\,dx.$$

By induction, assume that the formula

$$(-\lambda)^n F_j(\lambda) = \int_{-\infty}^{\infty} (L^n f)(x)\psi_j(x,\lambda)\,dx$$

holds. Then

$$\begin{aligned}(-\lambda)^{n+1} F_j(\lambda) &= \int_{-\infty}^{\infty} (L^n f)(x) L\psi_j(x,\lambda)\,dx \\ &= \left[(L^n f)(x)\psi_j'(x,\lambda) - \frac{d}{dx}(L^n f)(x)\psi_j(x,\lambda)\right]\Bigg|_{-\infty}^{\infty} \\ &\quad + \int_{-\infty}^{\infty} (L^{n+1} f)(x)\psi_j(x,\lambda)\,dx.\end{aligned}$$

and, by condition (3-iii) and the boundedness of $\psi_j(x,\lambda)$ and $\psi_j'(x,\lambda)$ as $x \to \pm\infty$ (see the beginning of Section 2), we have

$$\left[(L^n f)(x)\psi_j'(x,\lambda) - \frac{d}{dx}(L^n f)(x)\psi_j(x,\lambda)\right]\Bigg|_{-\infty}^{\infty} = 0.$$

Hence,

$$(-\lambda)^{n+1} F_j(\lambda) = \int_{-\infty}^{\infty} (L^{n+1} f)(x)\psi_j(x,\lambda)\,dx,$$

and since $L^{n+1} f \in L_2(R)$, it follows that $\lambda^{n+1} F_j(\lambda) \in L_2(R, d\rho_j)$. ∎

The proof of the following Theorem is similar to the proof of Theorem 1, and therefore, is omitted.

THEOREM 2. *A function f is the transform* (35) *of functions $F_j \in L_2(R, d\rho_j)$, $j = 1, \ldots, m$, with compact support if and only if f satisfies conditions* (3-*i*)–(3-*iii*) *of Lemma* 3 *and*

$$\lim_{n\to\infty} \|L^n f\|_{L_2(R)}^{1/n} < \infty.$$

EXAMPLE 4 (The real Fourier transform). Consider the singular Sturm–Liouville problem on the whole line:

$$y'' = -\lambda y, \quad -\infty < x < \infty, \quad \text{with } |y(\pm\infty)| < \infty.$$

Since $q(x) = 0$ is even, we have case (C). It is easy to see that

$$m_1(\lambda) = i\sqrt{\lambda}, \qquad \psi_1(x,\lambda) = e^{-ix\sqrt{\lambda}},$$

$$m_2(\lambda) = -i\sqrt{\lambda}, \qquad \psi_2(x,\lambda) = e^{ix\sqrt{\lambda}}$$

$$d\xi(\lambda) = \frac{1}{2\pi\sqrt{\lambda}}d\lambda = \frac{ds}{\pi}, \qquad d\zeta(\lambda) = \frac{\sqrt{\lambda}}{2\pi}\,d\lambda = \frac{s^2}{\pi}\,ds, \quad \text{on } (0,\infty),$$

where $\lambda = s^2$. The transforms (27), (28), and (33) become

$$\begin{aligned} E(\lambda) &= \int_{-\infty}^{\infty} \cos sx\, f(x)\, dx, \quad F(\lambda) = -\int_{-\infty}^{\infty} \frac{\sin sx}{s} f(x)\, dx, \\ f(x) &= \frac{1}{\pi}\int_0^{\infty} \cos sx E(\lambda)\, ds - \frac{1}{\pi}\int_0^{\infty} \frac{\sin sx}{s} F(\lambda) s^2\, ds. \end{aligned} \tag{36}$$

If we denote $E(\lambda)$, $-sF(\lambda)$, by $\widetilde{E}(s)$, $\widetilde{F}(s)$, respectively, then formulas (36) become

$$\begin{aligned} \widetilde{E}(s) &= \int_{-\infty}^{\infty} \cos sx\, f(x)\, dx, \quad \widetilde{F}(s) = \int_{-\infty}^{\infty} \sin sx\, f(x)\, dx, \\ f(x) &= \frac{1}{\pi}\int_0^{\infty} \cos sx\, \widetilde{E}(s)\, ds + \frac{1}{\pi}\int_0^{\infty} \sin sx\, \widetilde{F}(s)\, ds, \end{aligned} \tag{37}$$

and the Parseval equation (30) takes the form

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{\pi}\int_0^{\infty} |\widetilde{E}(s)|^2\, ds + \frac{1}{\pi}\int_0^{\infty} |\widetilde{F}(s)|^2\, ds.$$

For the real Fourier transform (37) it holds

COROLLARY 4. *A function f is the real Fourier transform* (37) *of functions $\widetilde{E}, \widetilde{F} \in L_2(R^+)$ with compact support if and only if, for any $n = 0, 1, 2, \ldots,$*

$$\begin{aligned} &f \text{ is infinitely differentiable on } R; \\ &f^{(n)}(x) \in L_2(R); \\ &\lim_{x\to\pm\infty} f^{(n)}(x) = 0; \\ &\lim_{n\to\infty} \|f^{(n)}(x)\|_{L_2(R)}^{1/n} < \infty. \end{aligned}$$

EXAMPLE 5 (The Hankel transform). Consider the Bessel differential equation again, but on the half line $(0, \infty)$:

$$\begin{aligned} y'' - \frac{\nu^2 - 1/4}{x^2} y = -\lambda\, y, \quad x \in (0, \infty), \quad \nu \geq 1, \\ |y(0)| < \infty, \quad |y(\infty)| < \infty. \end{aligned} \tag{38}$$

Since $q(x) = \frac{\nu^2 - 1/4}{x^2}$ has a singularity at $x = 0$, the problem is equivalent to a singular Sturm–Liouville problem on the whole line. As usual, let $\lambda = s^2$. In this case,

$$\psi_1(x, \lambda) = \frac{\sqrt{x}\, J_\nu(xs)}{\sqrt{a}\, J_\nu(as)}, \qquad 0 < a < \infty,$$

and

$$d\xi(\lambda) = \begin{cases} \frac{a}{2} J_\nu^2(as)\, d\lambda, & \lambda > 0, \\ 0, & \lambda < 0. \end{cases}$$

Therefore, formulas (31) and (32) take the form

$$G(\lambda) = \int_0^\infty \frac{\sqrt{x}\, J_\nu(xs)}{\sqrt{a} J_\nu(as)} f(x)\, dx,$$

$$f(x) = \int_0^\infty \sqrt{ax} J_\nu(as) J_\nu(xs) G(\lambda)\, s\, ds.$$

By putting $G(\lambda)\sqrt{a} J_\nu(as) = F(s)$, we have

$$\begin{aligned} F(s) &= \int_0^\infty \sqrt{x} J_\nu(sx) f(x)\, dx, \\ f(x) &= \int_0^\infty \sqrt{x} J_\nu(sx) s F(s)\, ds, \end{aligned} \tag{39}$$

which is a pair of Hankel transforms. Since the transform (39) has the form (11), Lemma 1 should be used, with the condition (1-iii′) instead of (1-iii). We show that for the Hankel transform condition (1-iii′) could be replaced by an easier verified condition:

$$\frac{d}{dx}\left(\frac{d^2}{dx^2} - \frac{\nu^2 - 1/4}{x^2}\right)^n f(x) = o(x^{-1/2-\nu}), \qquad x \to 0+. \tag{40}$$

In fact, let $s^n F(s) \in L_2(R^+)$. Then $s^n F(s) \in L_1(R^+)$. Since $\frac{d}{dx}[\sqrt{x} J_\nu(xs)] = O(x^{\nu-1/2})$ (see [1]) we have

$$\begin{aligned} \frac{d}{dx}\left(\frac{d^2}{dx^2} - \frac{\nu^2 - 1/4}{x^2}\right)^n f(x) &= \int_0^\infty \frac{d}{dx}[\sqrt{x} J_\nu(xs)] s^{2n} F(s)\, ds \\ &= O(x^{\nu-1/2}) = o(x^{-1/2-\nu}). \end{aligned}$$

Let (40) hold. Then

$$\left(\frac{d^2}{dx^2} - \frac{\nu^2 - 1/4}{x^2}\right)^n f(x) = o(x^{1/2-\nu}).$$

Because [1] $\sqrt{x} J_\nu(xs) = O(x^{\nu+1/2})$, we have $W_x((L^n f)(x), \sqrt{x} J_\nu(xs)) = o(1)$, and condition (1-iii′) is fulfilled.

Hence, we arrive at

COROLLARY 5. *Let* $L = \frac{d^2}{dx^2} - \frac{\nu^2 - 1/4}{x^2}$. *A function* f *is the Hankel transform* (39) *of a function* $F \in L_2(R^+)$ *with compact support if and only if condition* (40) *is satisfied, and*

$$f \text{ is infinitely differentiable on } R^+; \tag{41}$$

$$L^n f \in L_2(R^+) \tag{42}$$

$$\lim_{x\to\infty} (L^n f)(x) = \lim_{x\to\infty} \frac{d}{dx}(L^n f)(x) = 0; \tag{43}$$

$$\lim_{n\to\infty} \|L^n f\|_{L_2(R^+)}^{1/n} < \infty. \tag{44}$$

This result was first established in [15].

EXAMPLE 6. Consider the previous Sturm–Liouville problem (38), but with $0 \le \nu < 1$. In this case

$$\psi_1(x, \lambda) = \sqrt{\frac{x}{a}} \frac{cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)}{cJ_\nu(as) - s^{2\nu} J_{-\nu}(as)},$$

$$d\xi(\lambda) = a \frac{[cJ_\nu(as) - s^{2\nu} J_{-\nu}(as)]^2}{c^2 - 2cs^{2\nu} \cos \nu\pi + s^{4\nu}} s\, ds,$$

for some $c < 0$. With some calculations as before we get a pair of transforms

$$G(\lambda) = \int_0^\infty \sqrt{\frac{x}{a}} \frac{cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)}{cJ_\nu(as) - s^{2\nu} J_{-\nu}(as)} f(x)\, dx,$$

$$f(x) = \int_0^\infty \sqrt{ax} \frac{[cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)][cJ_\nu(as) - s^{2\nu} J_{-\nu}(as)]}{c^2 - 2cs^{2\nu} \cos \nu\pi + s^{4\nu}} G(\lambda) s\, ds.$$

By putting $F(s) = G(\lambda)\sqrt{a}[cJ_\nu(as) - s^{2\nu} J_{-\nu}(as)]$ we obtain

$$F(s) = \int_0^\infty \sqrt{x}[cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)] f(x)\, dx,$$

and

$$f(x) = \int_0^\infty \sqrt{x} \frac{cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)}{c^2 - 2cs^{2\nu} \cos \nu\pi + s^{4\nu}} F(s)\, s\, ds. \tag{45}$$

As $x \to 0+$ we have [1]

$$\sqrt{x}[cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)] = O(x^{1/2-\nu}),$$

$$\frac{d}{dx}\left\{\sqrt{x}[cJ_\nu(xs) - s^{2\nu} J_{-\nu}(xs)]\right\} = O(x^{-1/2-\nu});$$

hence, by Remark 1, the condition (1-iii′) could be replaced by

$$\left(\frac{d^2}{dx^2} - \frac{\nu^2 - 1/4}{x^2}\right)^n f(x) = o(x^{\nu+1/2}), \tag{46}$$

$$\frac{d}{dx}\left(\frac{d^2}{dx^2} - \frac{\nu^2 - 1/4}{x^2}\right)^n f(x) = o(x^{\nu-1/2}), \tag{47}$$

as $x \to 0+$, and we have

COROLLARY 6. *A function f is the transform* (45) *of a function $F(s) \in L_2(R^+, s\,ds)$ with compact support if and only if f has the properties* (41)–(43), (46)–(47), *and* (44).

Here we use the fact that on a finite interval $[0, \delta]$ the weight $s[c^2 - 2cs^{2\nu}\cos\nu\pi + s^{4\nu}]^{-1}$ and s are equivalent.

EXAMPLE 7 (The Kontorovich–Lebedev transform). Consider the Sturm–Liouville problem

$$y'' - e^{2x}y = -\lambda y, \quad -\infty < x < \infty. \tag{48}$$

By putting $z = e^x$, $\lambda = s^2$, Eq. (48) can be transformed into

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (s^2 - z^2)y = 0. \tag{49}$$

By comparing (49) with the differential equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2 + \nu^2)y = 0,$$

whose solutions are the Bessel functions of the third kind [1] $K_\nu(z)$ and $I_\nu(z)$, we conclude that $K_{is}(e^x)$ and $I_{-is}(e^x)$ are the solution to (48) that belong to $L_2(0, \infty)$ and $L_2(-\infty, 0)$, respectively. Thus

$$\psi_1(x, \lambda) = \frac{I_{-is}(e^x)}{I_{-is}(1)}, \qquad \psi_2(x, \lambda) = \frac{K_{is}(e^x)}{K_{is}(1)},$$

$$W(\psi_1, \psi_2) = m_1(\lambda) - m_2(\lambda) = -\frac{1}{I_{-is}(1)K_{is}(1)}.$$

Since K_{is} is an even function in s, and I_{-is} has a branch point at $s = 0$, we get case (B), and the expansion theorem takes the form [11]:

$$\tilde{f}(x) = \frac{1}{\pi i}\left\{\int_0^\infty K_{is}(e^x)\,d\lambda\int_{-\infty}^x [I_{-is}(e^y) - I_{is}(e^y)]\tilde{f}(y)\,dy\right.$$
$$\left. + \int_0^\infty [I_{-is}(e^x) - I_{is}(e^x)]\,d\lambda\int_x^\infty K_{is}(e^y)\tilde{f}(y)\,dy\right\}.$$

Putting $u = e^x$, $f(u) = \tilde{f}(x)$, we obtain the pair of Kontorovich–Lebedev transforms

$$\begin{aligned} F(s) &= \int_0^\infty K_{is}(u)\frac{f(u)}{u}\,du, \\ f(u) &= \frac{2}{\pi^2}\int_0^\infty s\sinh\pi s\,K_{is}(u)F(s)\,ds. \end{aligned} \tag{50}$$

In the new variable u the operators $(\frac{d^2}{dx^2} - e^{2x})^n$ and $\frac{d}{dx}(\frac{d^2}{dx^2} - e^{2x})^n$ take the forms $(u^2\frac{d^2}{du^2} + u\frac{d}{du} - u^2)^n$ and $u\frac{d}{du}(u^2\frac{d^2}{du^2} + u\frac{d}{du} - u^2)^n$, respectively. The condition as $x \to -\infty$ becomes a condition as $u \to 0+$, and

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^2\, dx = \int_0^{\infty} |f(u)|^2 \frac{du}{u}.$$

Consequently, Theorem 2 for the Kontorovich–Lebedev transform (50) takes the form

COROLLARY 7. *Let* $L = u^2\frac{d^2}{du^2} + u\frac{d}{du} - u^2$. *A function* f *is the Kontorovich–Lebedev transform* (50) *of a function* $F(s) \in L_2(R^+; s^2\, ds)$ *with compact support if and only if, for any* $n = 0, 1, 2, \ldots,$

$$f \text{ is infinitely differentiable on } R^+;$$

$$L^n f \in L_2(R^+);$$

$$\lim_{u\to 0}(L^n f)(u) = \lim_{u\to 0} u\frac{d}{du}(L^n f)(u) = 0;$$

$$\lim_{u\to \infty}(L^n f)(u) = \lim_{u\to \infty} u\frac{d}{du}(L^n f)(u) = 0;$$

$$\lim_{n\to\infty} \|(L^n f)(u)\|^{1/n}_{L_2(R^+;u^{-1})} < \infty.$$

Here we use the fact that on a finite interval $[0, \delta]$, the weights $s \sinh \pi s$ and s^2 are equivalent.

EXAMPLE 8 (The Jacobi transform). Let

$$y'' - q(x)y = -\lambda y, \qquad 0 < x < \infty, \tag{51}$$

where

$$q(x) = \frac{2(\sigma - 1)(2\gamma - \sigma - 1)\cosh x + 2\sigma^2 - 4\gamma\sigma + (1 - 2\gamma)^2}{4\sinh^2 x}.$$

This equation can be obtained from the hypergeometric differential equation

$$X(1 - X)Y'' + [\gamma - (\sigma + 1)X]Y' - \lambda^* Y = 0, \tag{52}$$

by putting

$$\lambda^* = \lambda + \frac{\sigma^2}{4}, \quad X = \sinh^2 \frac{x}{2}, \quad \text{and} \quad Y = r(x)y, \tag{53}$$

where

$$r(x) = \left(\frac{e^x - 1}{e^x + 1}\right)^{1/2+\sigma/2-\gamma} \sinh^{-\sigma/2} x.$$

The solutions to (52) are [1]

$$Y_1 = F(a, b; c; X), \qquad Y_2 = X^{1-c} F(a - c + 1, b - c + 1, 2 - c; X),$$

where $c = \gamma$, $a + b = \sigma$, $ab = \lambda^*$. Solving for a and b we get

$$a = \frac{\sigma}{2} + i\sqrt{\lambda}, \qquad b = \frac{\sigma}{2} - i\sqrt{\lambda},$$

which yields

$$Y_1(X) = F\left(\frac{\sigma}{2} + i\sqrt{\lambda}, \frac{\sigma}{2} - i\sqrt{\lambda}; \gamma; X\right)$$

as a solution to (52). If we put

$$\gamma = \alpha + 1, \qquad \sigma = \alpha + \beta + 1, \qquad X = \frac{1 - Z}{2},$$

we find that

$$Y_1\left(\frac{1-z}{2}\right) = F\left(\frac{\sigma}{2} + i\sqrt{\lambda}, \frac{\sigma}{2} - i\sqrt{\lambda}; \alpha + 1; \frac{1-z}{2}\right)$$

is a solution to the differential equation

$$(1 - Z^2)Y'' + [(\beta - \alpha) - (\alpha + \beta + 2)Z]Y' - \left(\frac{\sigma^2}{4} + \lambda\right)Y = 0,$$

which is the Jacobi differential equation. Let $\gamma > 2$ so we have a limit point case. Setting $Z = \cosh x$, we obtain a modified Jacobi function

$$\tilde{\varphi}_\lambda^{(\alpha,\beta)}(x) = F\left(\frac{\sigma}{2} + i\sqrt{\lambda}, \frac{\sigma}{2} - i\sqrt{\lambda}; \alpha + 1; -\sinh^2\frac{x}{2}\right).$$

Then, by (53) it follows that

$$\begin{aligned}\psi_1(x, \lambda) &= \frac{1}{r(x)}\tilde{\varphi}_\lambda^{(\alpha,\beta)}(x) = \left(\frac{e^x + 1}{e^x - 1}\right)^{1/2 + \sigma/2 - \gamma} \sinh^{\sigma/2} x\,\tilde{\varphi}_\lambda^{(\alpha,\beta)}(x) \\ &= 2^{-(\alpha+\beta+1)/2}\left(\sinh\frac{x}{2}\right)^{\alpha+1/2}\left(\cosh\frac{x}{2}\right)^{\beta+1/2}\tilde{\varphi}_\lambda^{(\alpha,\beta)}(x)\end{aligned}$$

is a solution to (51). Similar expression can be obtained for ψ_2 in terms of Y_2. It can be shown [11] that

$$\begin{aligned}&\Im\left[\frac{\psi_2(x, \lambda)}{m_1(\lambda) - m_2(\lambda)}\right] \\ &\quad = -\frac{1}{2^{\sigma+2}\sqrt{\lambda}}\left|\frac{\Gamma(\sigma/2 + i\sqrt{\lambda})\Gamma(\alpha + 1 - \sigma/2 + i\sqrt{\lambda})}{\Gamma(\alpha + 1)\Gamma(2i\sqrt{\lambda})}\right|^2 \psi_1(x, \lambda).\end{aligned}$$

Therefore, we have

$$F(\lambda) = \int_0^\infty \psi_1(x, \lambda) f(x)\, dx,$$

and

$$f(x) = \frac{1}{2^{\sigma+2}\pi} \int_0^\infty \left| \frac{\Gamma(\sigma/2 + i\sqrt{\lambda})\Gamma(\alpha + 1 - \sigma/2 + i\sqrt{\lambda})}{\Gamma(\alpha+1)\Gamma(2i\sqrt{\lambda})} \right|^2 \times \psi_1(x, \lambda) F(\lambda) \frac{d\lambda}{\sqrt{\lambda}}, \tag{54}$$

and the Parseval formula takes the form

$$\int_0^\infty |f(x)|^2\, dx = \frac{1}{2^{\sigma+2}\pi} \times \int_0^\infty \left| \frac{\Gamma(\sigma/2 + i\sqrt{\lambda})\Gamma(\alpha + 1 - \sigma/2 + i\sqrt{\lambda})}{\Gamma(\alpha+1)\Gamma(2i\sqrt{\lambda})} \right|^2 |F(\lambda)|^2 \frac{d\lambda}{\sqrt{\lambda}}.$$

Because

$$\psi_1(x, \lambda) = O(x^{\gamma - 1/2}), \qquad \psi_1'(x, \lambda) = O(x^{\gamma - 3/2}), \qquad x \to 0+,$$

and $\gamma > 2$, Lemma 1 and Theorem 1 still hold for the Jacobi transform (54) if the condition (1-iii′) is replaced by

$$\lim_{x \to 0+} (L^n f)(x), \ \lim_{x \to 0+} \frac{d}{dx}(L^n f)(x) \text{ exist for any } n.$$

Finally, by setting

$$p = 2\gamma - 2\sigma - 2, \qquad q = 2\sigma - \gamma + 1, \qquad \rho = \sigma = \frac{p}{2} + q,$$

$$\Delta(x) = (e^x - e^{-x})^p (e^{2x} - e^{-2x})^q,$$

$$c(\lambda) = 2^{\rho - i\lambda} \frac{\Gamma\left(\frac{p+q+1}{2}\right)\gamma(i\lambda)}{\Gamma\left(\frac{\rho + i\lambda}{2}\right)\Gamma\left(\frac{p+q+1-\rho+i\lambda}{2}\right)},$$

$$\varphi_\lambda(x) = \tilde{\varphi}_{\frac{\lambda^2}{4}}(2x) = F\left(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \frac{p+q+1}{2}; -\sinh^2 x\right),$$

$$\tilde{f}(x) = 2^{(3-\sigma)/2}\sqrt{\pi}(\sinh x)^{-\alpha - 1/2} x (\cosh x)^{-\beta - 1/2} f(2x),$$

$$\widetilde{F}(\lambda) = F\left(\frac{\lambda^2}{4}\right),$$

we obtain the pair of the Jacobi transforms and the Parseval equation in the standard form [4]; see also [5]:

$$\widetilde{F}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_\lambda(x)\Delta(x) f(x)\, dx,$$

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \varphi_\lambda(x) |c(\lambda)|^{-2} |\widetilde{F}(\lambda)|^2\, d\lambda,$$

$$\int_0^\infty \Delta(x) |\tilde{f}(x)|^2\, dx = \int_0^\infty |c(\lambda)|^{-2} |\widetilde{F}(\lambda)|^2\, d\lambda.$$

Similar results can be obtained for the finite Jacobi transform considered in [22].

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